Why Do we Care about Feb 11, 2025 Orthogonality ? There are infinitely many reasons to care about orthogonalize, we only present one used in function approximation. Orthogonality helps us becare finding a best approximate will be equivalent to solving a big system of linear equations. This is very computationaly involved. But using orthogonal basis make these computation much more efficient.

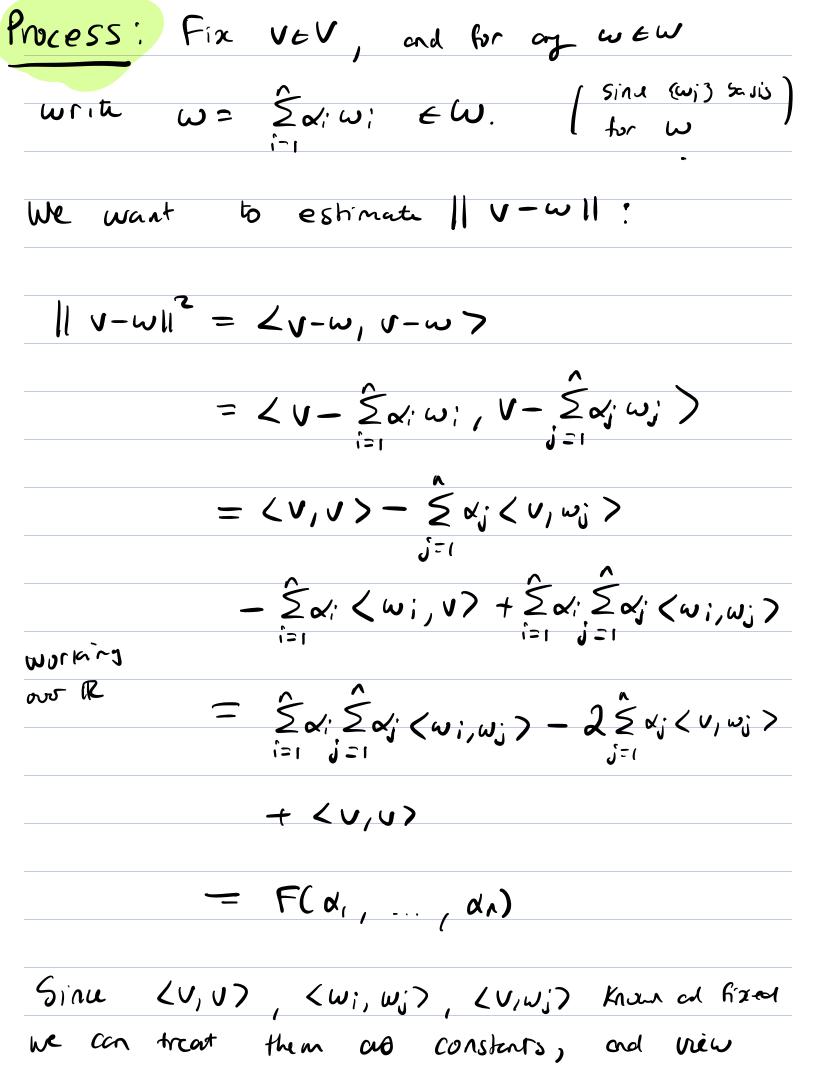
(toal: We are given a complicated finction P, and want to approximite / estimate fusing Simpler or setter understand finction.

Our setting is in Inner Product spaces, and we will say if is a good opproximation / cluse to f when 11f-f11 sml.

Setup: let V be on inner product space, one IR ord Ewi, wz, ..., w. 3 = V lineary independent, s.+ W= spon Ew1,.., wn3 a n-dim vector spon. The complicated function of lives in V, and we want to find few to is a "good" approximation for f. The vectors  $\Xi w_{1,...,w_n} = 3$  with always I known and " easy to indested" functurs.

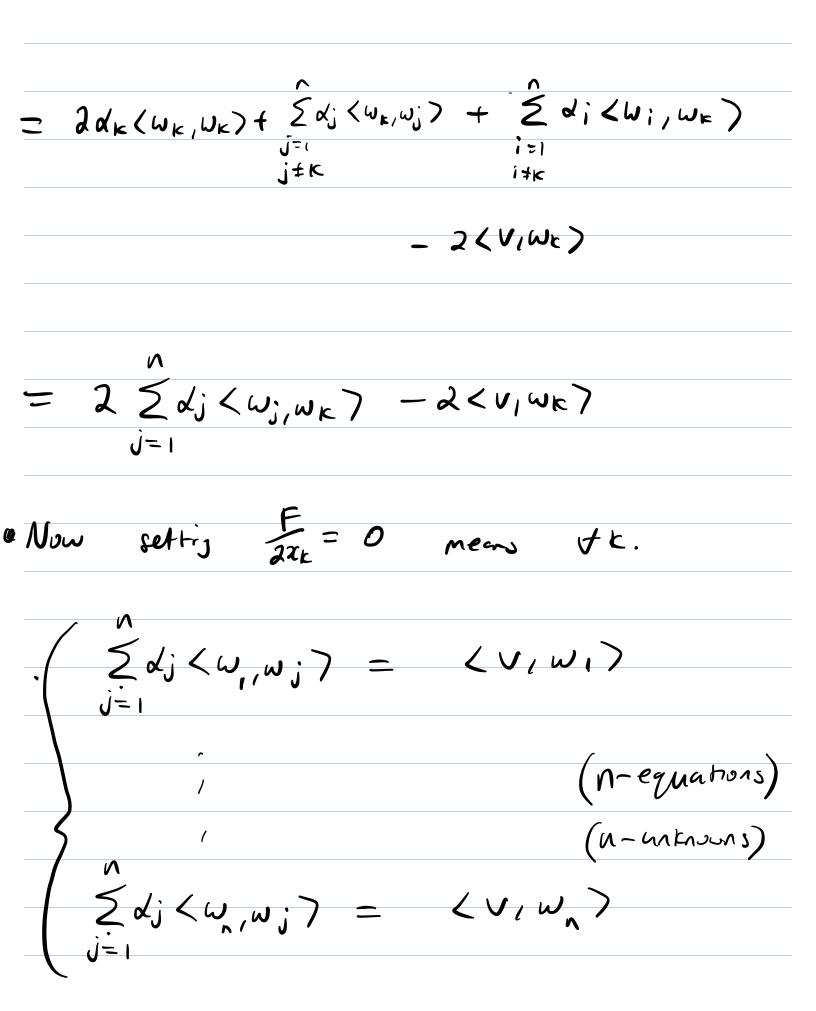
We say FEW is a best approximation to f wrt w to mean  $\|f-\widetilde{f}\| = \inf_{w \in \omega} \|f-w\|.$ 

@ One car show that in inner-product spaces, ve alwage have a migue best opposimilie (Friedery - Insel - Sporce, section 6.2)



 $F: R^{-2}R$ as a polynomial in x11. , xn. Recall we want to Minimize 11-w11<sup>2</sup>. That is we want to Mininize F. So set 2F = 0 , VK and solve 22K to find critical points. Now observe that Calculate 2F a get  $\frac{\partial}{\partial \alpha_{k}} \left( \hat{\Sigma} \hat{\Sigma} \alpha_{i} \alpha_{j} \langle \omega_{i}, \omega_{j} \rangle - 2 \hat{\Sigma} \alpha_{j} \langle v, \omega_{j} \rangle + \langle v, v \rangle \right)$  $= \sum_{i=1}^{n} \frac{2}{2\alpha_{k}} \left( \sum_{j=1}^{n} \frac{2}{\alpha_{i}} \left( \sum_{j=1}^{n} \frac{2}{\alpha_{i}} \left( \sum_{j=1}^{n} \frac{2}{\alpha_{j}} \right) \right) \right) - 2 \left( \sum_{j=1}^{n} \frac{2}{\alpha_{j}} \right) \right) \right) \right) \right)$  $= \underbrace{\sum_{i=1}^{2} \frac{1}{2\alpha_{k}} \left( d_{i} \underbrace{\sum_{j=1}^{2} d_{j} \langle \omega_{i}, \omega_{j} \rangle}_{j=1} - \frac{1}{2\langle v_{i}, \omega_{k} \rangle} - \frac{1}{2\langle v_{i}, \omega_{k} \rangle}_{i\neq k} \right)}_{i\neq k}$  $= \frac{2}{2\alpha\kappa} \left( \frac{2}{\alpha\kappa} \left( \frac{2}{\omega\kappa} \left( \frac{2}{\omega} \left( \frac{2}{\omega} \left( \frac{2}{\omega} \left( \frac{2}{\omega} \left( \frac{2}{$ 

 $-2\langle v, u_k \rangle$ 



But this can be represented as a Matrix sol:  $= \sum \left[ \langle w_{k}, w_{1} \rangle \langle w_{k}, w_{2} \rangle \cdots \langle w_{k}, w_{n} \rangle \right] \left[ \begin{pmatrix} \alpha_{1} \\ \kappa_{2} \\ \vdots \\ \vdots \\ \end{pmatrix} \right]$ =  $\langle v_1 \omega_k \rangle$ the above tells us what the kth now of our matrix looks like. 1.e  $\langle w_{1}, w_{1} \rangle \langle w_{1}, w_{2} \rangle = - \cdot \langle w_{1}, w_{n} \rangle$  $\langle w_{2}, w_{1} \rangle \langle w_{2}, w_{2} \rangle$ α<sub>2</sub> - <ν, ω<sub>2</sub>? - Zwn, Waz dn (Vin) < wr, w,7 X ( Symmetrie if Va Ruspa, conj-symmetrie if van () Where we call the matrix above the Gran matrix associated to the basis we

· Now find (d1,..,dn) that minimize  $F(d_1, \ldots, d_n)$ . · So far: Solving \* gives us access to finding the best opproxime, since the Sest opproximate is Just  $w_{o} = \sum_{i=1}^{n} \widetilde{\alpha}_{i} w_{i} \quad w_{i} \quad \omega_{i} \quad \widetilde{\alpha}_{i_{1} \cdots i_{n}} \widetilde{\alpha}_{n}$ Solution to D. Now @ is very computationly hea Solving Systems of equation when the matrix is diagonal officient 1 Computationly heavy. efficient! is much more So if we started with Ewi,..., wh? on Orthogond set, the the Gran matrix would look like.

<wi, w, >  $(\omega_2, \omega_27)$ ł ι (wn, Wn) Practical Example yet to cone!